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BIFURCATION OF THE LINES OF AN INCOMPRESSIBLE VISCOUS-FLUID FLOW IN A RECTANGULAR CAVITY WITH A MOVING WALL

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The bifurcation of the lines of a viscous-fluid flow in a rectangular cavity with a moving cover has been investigated for different ratios between the sides of the cavity and different Reynolds numbers on the basis of the qualitative theory of dynamic systems. The critical parameters of the problem, at which the type of singular points changes, and other topological characteristics of a vortex flow in the indicated cavity have been determined and the corresponding bifurcation diagrams have been constructed.

Introduction. The study of the stability of two- and three-dimensional vortex flows of a viscous fluid is one of the fundamental problems of hydrodynamics, which concerns the problems of control of separation flows. The topological characteristics of a flow can be determined on the basis of solution of the Navier–Stokes equations. However, the number of problems that can be exactly solved in this way is limited; this being so, numerical methods are used in the majority of cases.

The problem on the flow of an incompressible viscous fluid in a rectangular cavity with a moving wall is a classical fluid-mechanics problem with closed boundaries. The main structural peculiarities of this flow are characteristic of other separation flows having a more complex geometry; therefore, solution of the problem on the indicated flow is used for testing and comparison of different numerical methods of integrating the Navier–Stokes equations.

Comprehensive data on the vortex structure and characteristics of a flow in a rectangular cavity are given in the literature [1, 2]. On the basis of systematization and analysis of these data, criteria have been developed for estimating the quality of the discrete model used [2] (the maximum value of the stream function in a computational region, the vertical and horizontal sizes of the secondary corner vortices, the coordinates and types of singular points).

The above-indicated problem is exactly solved on the basis of Navier–Stokes equations only in the case of the Stokes regime of flow ($\mathrm{Re} \to 0$) [3]. In the other cases, numerical methods are used. Navier–Stokes equations can be numerically integrated using the SIMPLE method and different modifications of the Leonard scheme [1]. The computational methodology used in [2] is based on solution of the Navier–Stokes equations by the finite-volume method within the framework of the scheme of splitting by physical processes with the use of the SIMPLEC pressure-correction procedure. The method of incomplete factorization of a matrix is used for solving the system of difference equations. In [4], the accuracies of different finite-difference schemes of discretization of convective flows were compared, and the authors of [5] and [6] proposed to increase the accuracy of numerical calculations by using the pseudospectral method and the Boltzmann-grid method.

In [7], a flow in a rectangular cavity was investigated for different ratios between the cavity sides and different velocities of travel of the upper and lower walls. The finite-element method with bunching of nodes of a grid in the neighborhood of local stagnation points was used for discretization of Navier–Stokes equations. The investigations were carried out for fairly small Reynolds numbers ($\mathrm{Re} < 100$). Reasonably exact results were not obtained by the finite-element method because it, when used for solving fluid-mechanics problems, provides a lower accuracy than the finite-difference method.

A direct numerical simulation of the transformation of a laminar flow in a square cavity into a chaotic motion has been carried out in [8]. Navier–Stokes equations were discretized by the MAC method on a uniform diversified grid. A stationary flow was transformed into a periodic motion through the Hopf bifurcation arising at $\mathrm{Re} \sim 7400$ [9].

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Despite the existence of a fairly large amount of various data that can be used for both the comparison of different schemes of solving the Navier–Stokes equations and the realization of the physical sense of a separation flow formed in a rectangular cavity, the topological characteristics of a vortex flow in this cavity are not completely understood.

In the present work, we investigated the bifurcation of the lines of an incompressible viscous-fluid flow in a rectangular cavity and determined the topological characteristics of this flow depending on the ratio between the sides of the cavity and the Reynolds number with the use of the qualitative theory of dynamic systems. The critical parameters of the problem, at which the type of singular points changes, were determined and bifurcation diagrams were constructed on the basis of numerical-simulation data.

Formulation of the Problem and Numerical Method. Let us consider an unsteady isothermic flow of an incompressible viscous fluid in a rectangular cavity with a ratio between the sides $H = L_1/L_2$, where L_1 and L_2 are the height and width of the cavity respectively (it is assumed that $L_1 < L_2$ and $H < 1$). This flow is induced by the movement of the upper wall of the cavity with a constant velocity $U = 1$ m/sec along the x axis. The values of the Reynolds number along the length of the cavity are calculated: $\mathrm{Re} = \rho UL/\mu$. Calculations are performed for a fluid with a density $\rho = 1.2$ kg/m^3 and a molecular viscosity μ corresponding to a definite Reynolds number.

The Navier–Stokes equations are written in physical variables. It is assumed that, at the initial instant of time, the fluid in the cavity is at rest ($u = v = 0$, $p = 10^5$ Pa). The boundary conditions of adhesion and nonpercolation are set at the cavity walls.

The Navier–Stokes equations are numerically solved by the method of splitting by physical processes on a nonuniform staggered grid [10] (projection method). The pressure at the center of a cell and the velocity components at its boundaries are determined. The points at which velocity components are calculated are found at the center of the line connecting neighboring nodes at which the pressure is determined. Derivatives are discretized with respect to time by the Adams–Bashfort scheme of second order of accuracy. Convective flows are discretized by the SMART scheme [11]. Diffusion flows are discretized using centered finite-difference formulas of the second order of accuracy. The Poisson equation for pressure is solved by the method of biconjugate gradients with stabilization (BiCGStab) [12].

For bunching of grid nodes, the following transformation of coordinates is done:

$$x_i = \frac{(\beta + 1)\,[(\beta + 1)/(\beta - 1)]^{2\bar{x}_i - 1} - (\beta - 1)}{2\,[1 + (\beta + 1)/(\beta - 1)]^{2\bar{x}_i - 1}}\,,$$

where $\beta = (1 - \delta)^{-1/2}$ and $\bar{x}_i = i/n$ for $i = 0, ..., n$. At $\delta \to 0$, nodes bunch near the walls of the cavity, and, at $\delta \to 1$, nodes bunch at its center.

For resolution of the structure of the flow in the neighborhood of singular points, grid nodes are bunched along the y coordinate depending on the longitudinal velocity component [7] estimated in each iteration along the grid line i. The neighboring nodes, at which the signs of the longitudinal velocity component are different, are determined. The transverse coordinate of the singular point y_s, at which $u = 0$, is determined using the linear interpolation

$$y_s = y_j + \frac{|u_j|}{|u_j| + |u_{j+1}|}\,(y_{j+1} - y_j)\,.$$

Here, u_j and u_{j+1} are the longitudinal-velocity components opposite in sign at the nodes j and $j + 1$ lying on the grid line i.

Then the nodes at which the transverse velocity component changes its sign are determined, after which the longitudinal coordinate of the singular point x_s, at which $v = 0$, is determined using the linear interpolation

$$x_s = x_i + \frac{|v_i|}{|v_i| + |v_{i+1}|}\,(x_{i+1} - x_i)\,.$$

Here, v_i and v_{i+1} are the transverse-velocity components opposite in sign at the nodes i and $i + 1$ lying on the grid line j.

TABLE 1. Results of Calculations of a Flow in a Square Cavity at Re = 100

Vortex	Grid	Vortex Center	ψ_{max}
M	81×81	(0.6115,0.7365)	0.1030*
M	101×101	(0.6139,0.7368)	0.1034*
M	129×129	(0.6172,0.7344)	0.1034**
L	81×81	(0.0327,0.0359)	$-1.19\cdot10^{-6}$*
L	101×101	(0.0318,0.0374)	$-1.64\cdot10^{-6}$*
L	129×129	(0.0313,0.0391)	$-1.75\cdot10^{-6}$**
R	81×81	(0.9368,0.0643)	$-1.12\cdot10^{-5}$*
R	101×101	(0.9422,0.0618)	$-1.59\cdot10^{-5}$*
R	129×129	(0.9453,0.0625)	$-1.25\cdot10^{-5}$**

*Calculation results of the present work.

**Data of [14].

Results of Calculations. The main purpose of our calculations was to determine the positions of critical points of a flow in a rectangular cavity and the problem parameters H and Re at which the streamlines bifurcate.

The streamlines were constructed using the system of equations determined by the parameters of the problem:

$$\frac{d\mathbf{r}}{dt} = \mathbf{v}\,(\mathbf{r}, \mathrm{Re}, H)\ .$$

The properties of the dynamic system were investigated using the qualitative theory of differential equations [13].

The flow in a rectangular cavity was calculated for Reynolds numbers Re = 0–1000 and different ratios between the cavity sides H = 0.125–1 on a grid containing 40,000 nodes. The numbers of nodes in the longitudinal and transverse directions were varied such that a plausible ratio between the sides of a computational cell was retained.

The results of numerical simulation of a flow in a rectangular cavity at H = 1 (square cavity) and Re = 0–1000 reflect the main features of a vortex flow in this cavity [2]. A flow in a rectangular cavity consists of a vortex occupying a larger part of the cavity and a number of corner vortices rotating in the opposite directions. At Re = 100, the primary vortex occupies the practically whole volume of the cavity and two secondary vortices are developed in the lower and upper parts of the flow in the lower part of the cavity.

The lines of flow in a square cavity (H = 1) at small Reynolds numbers (Re $\to$ 0) are practically symmetric relative to the geometric center of the cavity and the primary vortex occupies the whole its volume. The center of the vortex (center-type critical point) is found at the point with coordinates (0.5, 0.5) and a limiting cycle is realized in the system.

The quality of the numerical solution was estimated by the location of the main vortex M, the location of the secondary corner vortices L (at the left, upstream) and R (at the right, downstream) near the lower wall, and the maximum value of the stream function at the center of these vortices ψ_{max}. The results of numerical simulation performed on different grids at Re = 100 are compared with the corresponding data of [14] in Table 1. The lines of flow in a rectangular cavity are shown in Fig. 1.

The bifurcation diagrams illustrating the topological features of a vortex flow in a rectangular cavity at different Reynolds numbers are presented in Fig. 2.

In a rectangular cavity ($H < 1$), the longitudinal velocity component is much larger than the transverse one (the smaller the H, the larger this difference) and critical points lie on the line y = const. To determine the character of a critical point, it will suffice to determine the coordinates of the grid nodes at which the sign of the transverse velocity component changes. For example, at Re $\to$ 0 this change (bifurcation of a streamline) happens near the geometric center of the cavity at H_* = 0.385 (Fig. 2a), which leads to the formation of two new singular points. The center-type critical point (at H = 1) is transformed into a saddle point and two centers (at the left and at the right of the saddle point) that move along the line y = const when H further decreases. At $H > H_*$, a single singular point is stable

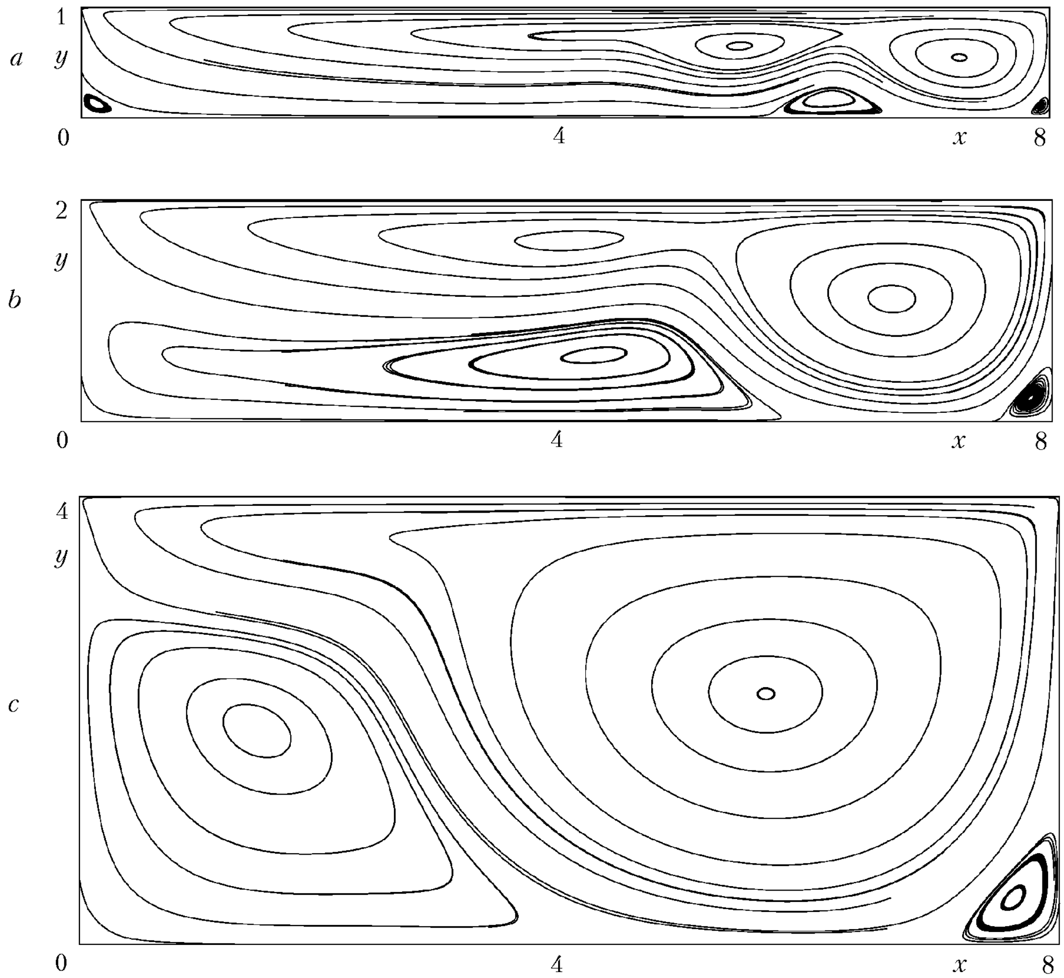

Fig. 1. Lines of flow in a rectangular cavity at Re = 500 and H = 0.125 (a), 0.25 (b), and 0.5 (c).

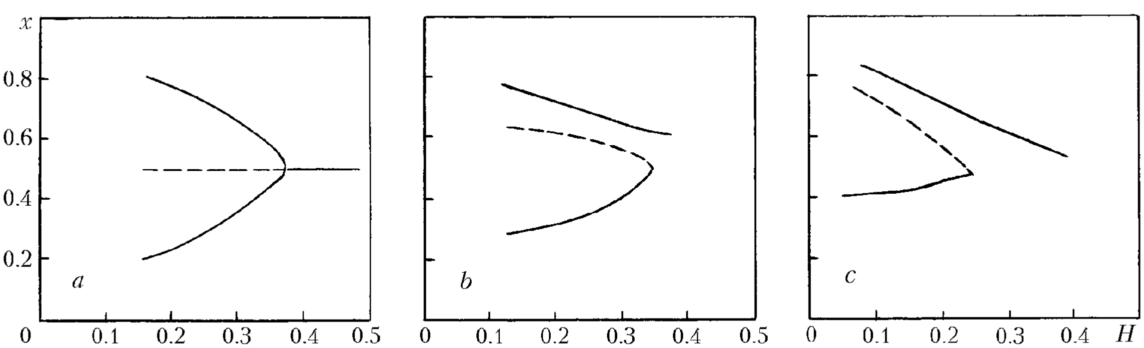

Fig. 2. Bifurcation diagrams at Re = 0 (a), 50 (b), and 100 (c).

and stationary, and at $H = H_*$ there arises an additional pair of singular points, each of which corresponds to two vortices into which the central vortex breaks down. As a result, a fork-type bifurcation arises and a pair of solutions obtained at the bifurcation point branches off with a zero initial amplitude that increases monotonically with increase in the supercriticality.

At Re > 0, the bifurcation of streamlines changes. A decrease in H at a fixed Reynolds number results in the fork-type bifurcation at Re = 0 being transformed into a bifurcation of the type of saddle–node bifurcation at Re > 0

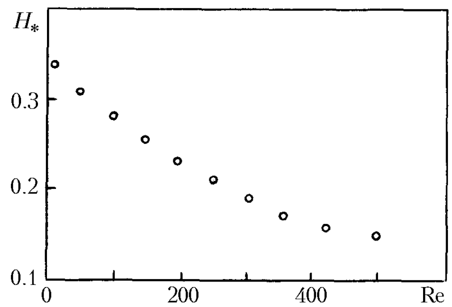

Fig. 3. Dependence of the critical ratio between the sides of a cavity on the Reynolds number.

(Fig. 2b and c). Investigation of the stream-function distribution along the line y = const (at H = 0.125 and Re = 100, critical points are found on the line y_s = 0.532) has shown that its minimum value corresponds to the saddle point and its maximum values corresponds to the centers.

The critical value of the ratio between the cavity sides, at which the type of singular points changes, decreases with increase in the Reynolds number (Fig. 3).

Conclusion. The dependence of the change in the type of singular points of a vortex flow (bifurcation of streamlines) in a rectangular cavity on the ratio between the cavity sides and on the Reynolds number has been investigated on the basis of numerical simulation of this flow. The data obtained can be used for determining the topological characteristics and features of separation flows in cavities of more complex geometries.

NOTATION

H, ratio between the sides of a cavity; L, size of a cavity side, m; n, number of grid nodes; p, pressure, Pa; **r**, radius-vector, m; Re, Reynolds number; t, time, sec; u, v, velocity components, m/sec; U, velocity of movement of the upper wall, m/sec; v, velocity vector, m/sec; x, y, space coordinates; β, δ, coefficients; μ, dynamic viscosity, kg/(m·sec); ρ, density, kg/m^3; ψ, stream function. Subscripts: max, maximum; s, critical point; *, critical ratio between the cavity sides.

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